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EQUIVALENT FORMULATIONS OF THE HIRSCH CONJECTURE FOR ABSTRACT POLYTOPES

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Prepared for:

Office of Naval Research
National Science Foundation

November 1972

ORC 72-28
NOVEMBER 1972

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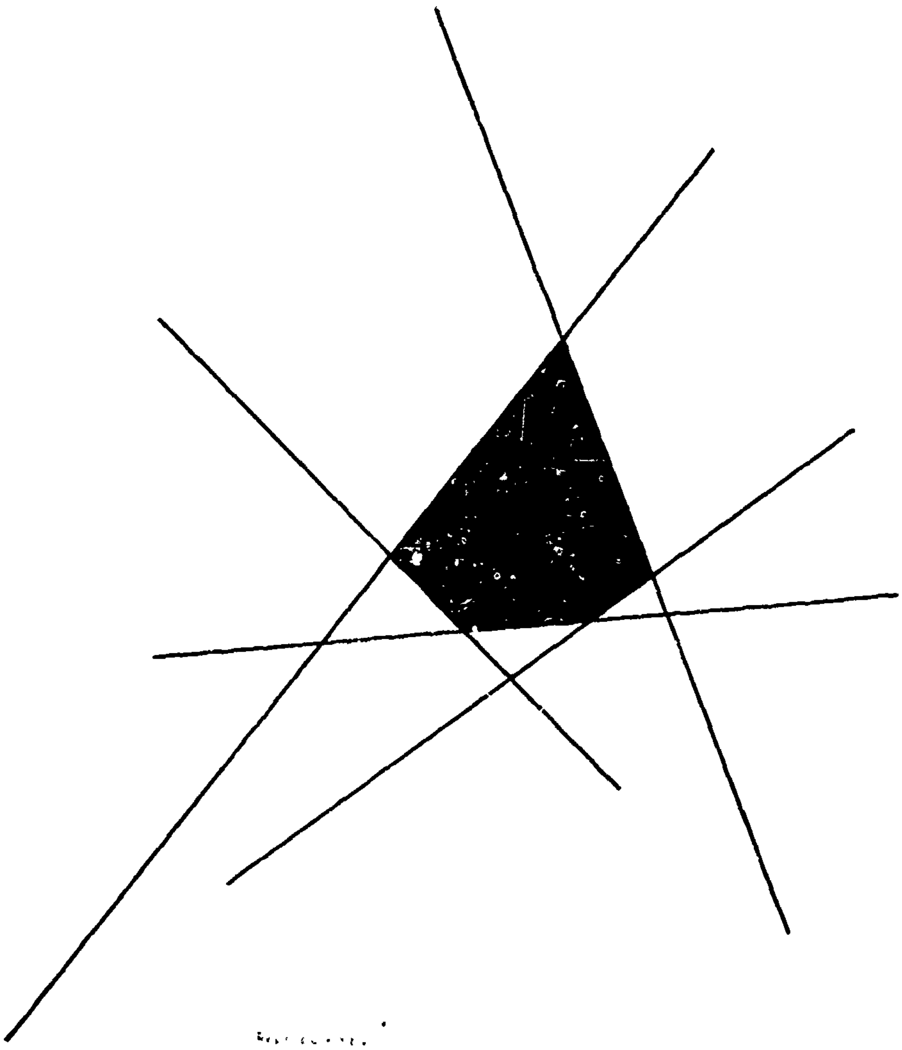

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This research has been partially supported by the Office of Naval Research under Contract N00014-69-A-0200-1010 and the National Science Foundation under Grant GK-23153 with the University of California.

I

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP

3. REPORT TITLE

EQUIVALENT FORMULATIONS OF THE HIRSCH CONJECTURE FOR ABSTRACT POLYTOPES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report

5. AUTHOR(S) (First name, middle initial, last name)

John A. Lawrence, Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1972	13/18	2

8a. CONTRACT OR GRANT NO. GK-23153	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 72-28
b. PROJECT NO.	
c	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES Also supported by the Office of Naval Research under Contract N00014-69-A-0200-1010.	12. SPONSORING MILITARY ACTIVITY NSF-GK-23153-Shephard The National Science Foundation Washington, D.C. 20550

13. ABSTRACT

SEE ABSTRACT.

II

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified
Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Polytopes						
Abstract Polytopes						
Hirsch Conjecture						

III

DD FORM 1 NOV 65 1473 (BACK)

Unclassified

Security Classification

A-31409

ABSTRACT

Abstract polytopes are mathematical creations which are defined by three axioms. It has been shown that simple polytopes are a proper subclass of abstract polytopes. Hence theorems proving facts about abstract polytopes in general, prove facts about simple polytopes in particular.

Klee and Walkup [2] showed the following four statements were mathematically equivalent for simple polytopes:

i) Any two vertices of a simple polytope can be joined by a W_v (nonreturning) path.

ii) $\Delta(n,d) \leq n - d$ (Hirsch conjecture).

iii) $\Delta(2d,d) \leq d$.

iv) For a Dantzig figure, (P,x,y) , $\delta_P(x,y) = d$.

The purpose of this paper is to show that the four statements above are equivalent for the larger class of abstract polytopes as well. Thus, it is possible to tackle the problem of the well-known Hirsch conjecture by applying the well defined structure and theorems of abstract polytopes to any of the above statements.

1. INTRODUCTION

In Adler [1], it was shown that the class of simple polytopes is a proper subclass of the class of abstract polytopes. Because of the simplicity and mathematical precision which define abstract polytopes, they become interesting and practical tools for investigating properties of simple polytopes.

Klee and Walkup [2], showed that the well known Hirsch conjecture is mathematically equivalent to three other statements. In this paper it will be shown that a corresponding set of statements are equivalent for abstract polytopes, and hence abstract polytope theory may shed new light in proving or disproving the Hirsch conjecture.

The method of proof will be the method of consecutive implication, i.e. for statements, i, ii, iii, iv, it is shown i $\Rightarrow$ ii $\Rightarrow$ iii $\Rightarrow$ iv $\Rightarrow$ i . Three implications are trivially proved. A constructive proof for the fourth implication is developed strictly from abstract polytope theory and is based on Klee and Walkup [2].

Finally an example of the construction is presented in the appendix.

2. NOTATIONS AND DEFINITIONS

Let C be a set of distinct symbols.

Any subset v, of d distinct symbols contained in C is called a vertex.

$$|v| \text{ denotes the number of symbols in } v .$$

Vertices u, v of d symbols are said to be neighbors iff $|u \cap v| = d - 1$.

A path from v_0 to v_k is a series of vertices $v_0, v_1, \ldots, v_k$ s.t. v_i, v_{i+1} are neighbors, $i = 0, \ldots, k-1$.

Definition of an Abstract Polytope

P is a labeled abstract polytope consisting of vertices $V(P)$ iff

(A1) $v \in V(P) \Rightarrow |v| = d$.

(A2) $v \in V(P) \Rightarrow$ for any symbol $T \in v$, $\exists$ exactly one other vertex $w \in V(P)$, s.t. $v - T \subset w$.

(A3) For any v_0, $v_k \in V(P)$, $\exists$ a path $R : v_0, v_1, \ldots, v_k$ s.t. $v_0 \cap v_k \subset v_i \;\; \forall \;\; v_i \in R$. ($R$ is called an A3 path.)

Note: We will use P also to denote the union of all symbols used in the vertices of $V(P)$.

$$P(n,d) \equiv \text{set of all abstract polytopes s.t. } |P| = n,\ |v| = d .$$

If $v \in V(P)$ and some set $S \subset v$ such that $|S| = k\ (> 0)$, S is said to generate a $d - k$ *face* on P; i.e., $F_P(S)$, the face generated by S on P, consists of *all* $v \in V(P)$ s.t. $S \subset v$. It is easily verified that after the k symbols common to $v \in V(F_P(S))$ have been dropped we have formed a new abstract polytope $Q \in \mathcal{P}(n',d-k)$ where $n' \leq n-k$. We say Q corresponds to $F_P(S)$ and the vertices of Q have a one to one correspondence to the vertices of $F_P(S)$, thus $q \in V(Q)$ corresponds to $q \cup S \in V(F_P(S))$.

$\delta_P(x,y) \equiv$ the *distance* from x to y on P, i.e., the length of the shortest path between x and y.

$$\Delta_a(n,d) \equiv \text{the max diameter of } \mathcal{P}(n,d)$$
$$\equiv \max_{P \in \mathcal{P}(n,d)} \ \max_{x,y \in V(P)} \delta_P(x,y) .$$

A *nonreturning (NR)* path from $v_0, v_1, \ldots, v_k$ on P is a path s.t. the symbol $T \in v_i$, $T \notin v_{i+1} \Rightarrow T \notin v_j$ $i+1 \leq j \leq k$.

A *Dantzig figure* (P,x,y) is an abstract polytope $P \in \mathcal{P}(2d,d)$ s.t. $\exists$ vertices $x, y \in V(P)$ s.t. $x \cap y = \varphi$.

3. THEOREM

The following four statements are equivalent for abstract polytopes:

i) $\exists$ an NR path between any two vertices x , $y \in V(P)$.

ii) $\Delta_a(n,d) \leq n - d$.

iii) $\Delta_a(2d,d) \leq d$.

iv) For every Dantzig figure (P,x,y) , $\delta_P(x,y) = d$.

Proof: (by consecutive implication)

(a) i => ii) if $\exists$ an NR path between x , $y \in V(P)$ then

$\delta_P(x,y) \leq n - (|x \cap y| + d)$. If $x \cap y = \phi$ result follows.

Say $x \cap y = S$, $|S| = k$. Consider $F_P(S)$. Then

$\delta_{F_P(S)}(x,y) \leq n' - (d - k) \leq n - k - (d - k) = n - d$.

$\therefore$ $\delta_P(x,y) \leq n - d$ $\forall$ $P \in \mathcal{P}(n,d)$.

(b) ii => iii) trivial. Let $n = 2d$.

(c) iii => iv) $\delta_P(x,y) \leq d$ by iii) . But since $x \cap y = \phi$, $\delta_P(x,y) \geq d$

$\therefore$ $\delta_P(x,y) = d$.

(d) iv => i)

Consider x , $y \in V(P)$. Assume $x \cap y = \phi$ for convenience. If $x \cap y = S$ $(\neq \phi)$ consider $F_P(S)$. After eliminating S from each vertex of $F_P(S)$ we have a new abstract polytope, Q , whose vertices correspond to $F_P(S)$. In particular the vertices which correspond to x and y have an empty intersection. Thus, we can apply the following theory to Q and make the corresponding transformation back to $F_P(S)$ and hence P .

Assume $n > 2d$. (If $n = 2d$ result follows trivially since $\delta_P(x,y) = d$.)

Notation

Let $M = P - (x \cup y)$; say $|M| = m$, i.e., $m = n - 2d$.

Form a set M' of new symbols s.t. $T \in M \Rightarrow T' \in M'$.

Now we form P^* by the algorithm, illustrated in Figure 1.

We now show $P^* \in (n + m, d + m)$. We show the more general fact:

Lemma 1:

$P^j \in \mathcal{P}(n + j, d + j)$.

Proof: (by induction)

Trivially $P^0 \in \mathcal{P}(n,d)$.

Suppose $P^{j-1} \in \mathcal{P}(n + j - 1, d + j - 1)$.

Consider P^j . It is easily verified A1 and A2 are satisfied.

To show A3, consider the following three cases. Suppose u^j , $v^j \in V(P^j)$.

Case I: $T' \in u^j$, $T' \in v^j$

$\exists$ u^{j-1} , $v^{j-1} \in V(P^{j-1})$ s.t. $u^{j-1} = u^j - T'$, $v^{j-1} = v^j - T'$.

Since $\exists$ an A3 path $u^{j-1} = u_0^{j-1}, \ldots, u_k^{j-1} = v^{j-1}$ on P^{j-1} .

Let $u_i^j = u_i^{j-1} \cup T'$. This is the required A3 path on P^j .

Case II: $T' \in u^j$, $T' \notin v^j$

By construction, $T' \notin v^j \Rightarrow T \in v^j$ and $\exists$ $y^j \in V(P^j)$ s.t. $y^j = v^j \cup T - T'$.

By Case I $\exists$ an A3 path, R , on P^j from u^j to y^j . $\therefore$ The required A3 path from u^j to v^j is R amended with the edge (y^j, v^j) .

Case III: $T' \notin u^j$, $T' \notin v^j$

Let y^j , $z^j \in V(P^j)$ be such that $y^j = u^j \cup T' - T$, $z^j = v^j \cup T' - T$.

Construction 1

Let $j = 0$, $P^0 = P$, $M^0 = M$, $M'^0 = M'$.

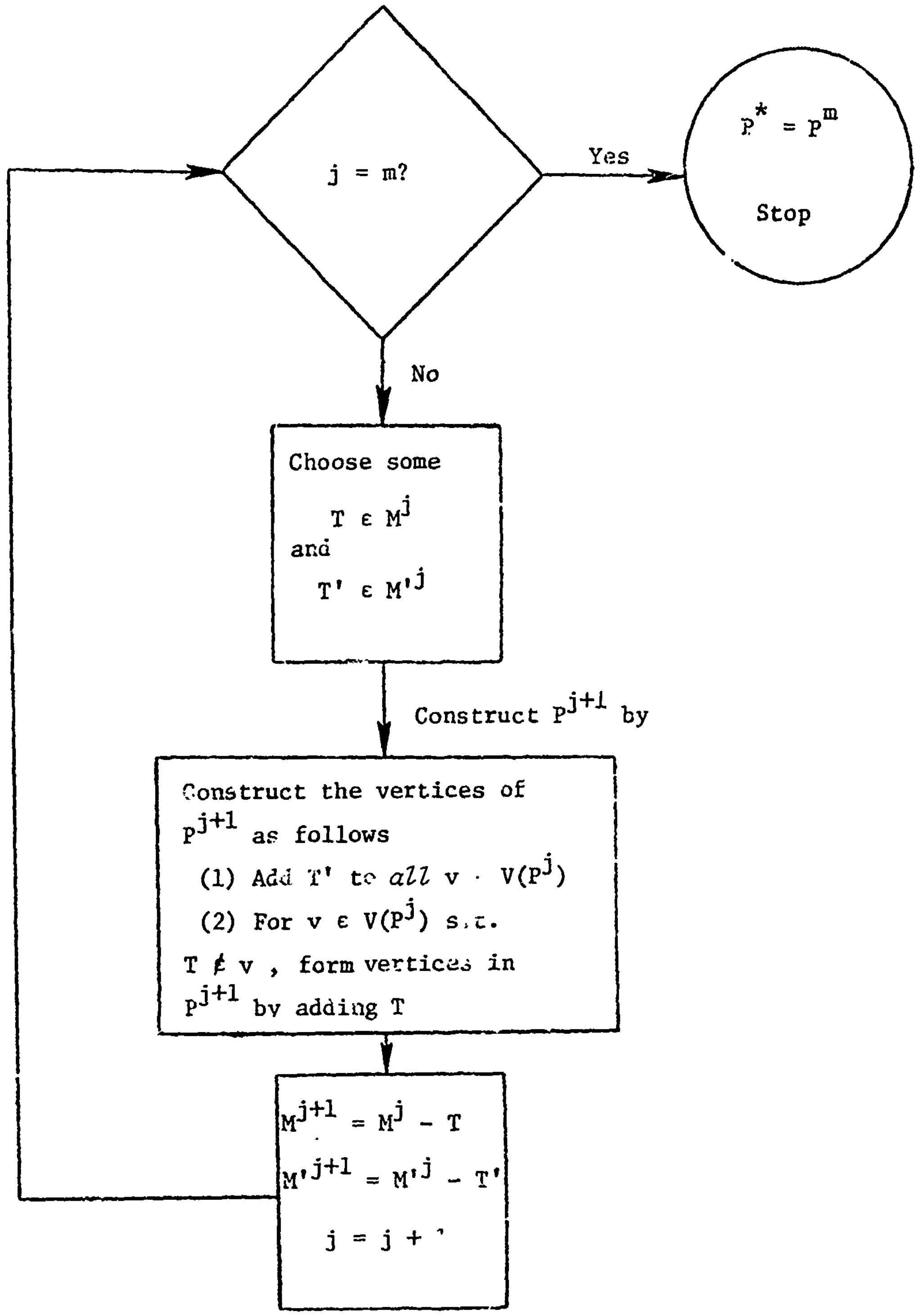


Figure 1

We know $\exists$ an A3 path R from y^j to z^j on P^j . $R : y^j = w_0^j, \ldots, w_f^j, \ldots, w_\ell^j, \ldots, w_k^j = z^j$ where for $f \le i \le \ell$, $T \in w_i^j$. (It is not necessary that $T \in w_i^j$ for any i or for only one interval.)

Consider the path $\bar{R} : u^j = \overline{w_0^j}, \ldots, \overline{w_f^j}, \ldots, \overline{w_\ell^j}, \ldots, \overline{w_k^j} = v^j$ s.t. for $0 \le i \le f - 1$ and $\ell + 1 \le i \le k$, $\overline{w_i^j} = w_i^j \cup T' - T$ and for $f \le i \le \ell$, $\overline{w_i^j} = w_i^j$.

It is not difficult to verify $\bar{R}$ exists and is the required A3 path from u^j to v^j on F^j . ■

Corollary 1:

$P^* \in P(n + m, d + m)$, i.e., $P^* \in (2(n - d), (n - d))$.

We now make the following observations:

Observation 1: $\exists$ nodes x^* , $y^* \in V(P^*)$ s.t. $x^* = x \cup M'$, $y^* = y \cup M$.

Observation 2: As noted in Lemma 1, for $w \in V(P^*)$, $T' \notin w \Rightarrow T \in w$ and $\exists$ a vertex $y \in V(P^*)$ s.t. $y = w \cup T' - T$.

Observation 3: P corresponds to $F_{P^*}(M')$.

Observation 4: (P^*, x^*, y^*) is a Dantzig figure. $\therefore$ Since $\delta_{P^*}(x^*, y^*) = d^*(=d + m)$, $\exists$ an NR path between x^* and y^* on P^* .

Notation

Let R^* be the NR path on P^* between x^* and y^* $\therefore$ $R^* : x^* = v_1$, $\ldots, v_{d^*} = y^*$.

Let $B'_{v_i} = M' \cap v_i$, i.e., all primed symbols in v_i .

$A'_{v_i} = M' - B'_{v_i}$.

$A_{v_i} = \{T \in M \mid T' \in v_i, T \notin v_i\}$.

Observation 5:

$$\forall\ v_i \in V(P^*)\ ,\ v_i = y_i \cup B'_{v_i} \cup A_{v_i} \quad \text{where} \quad y_i \in V(P)\ .$$

Construction 2:

Form a sequence of vertices $R' : w_1, \ldots, w_d^*$ s.t. $w_i = v_i \cup A'_{v_i} - A_{v_i}$, where $v_i \in R^*$.

Lemma 2:

i) $w_1 = x \cup M' \in F_{P^*}(M')$.

ii) $w_i \in F_{P^*}(M')$ $\forall_i$.

iii) $w_{d^*} = y \cup M' \in F_{P^*}(M')$.

iv) w_i , w_{i+1} are either neighbors or the same vertex, $\forall_i$.

v) If w_i , w_{i+1} are the same vertex, delete one of them from R' . Then R' is an NR path from $x \cup M'$ to $y \cup M'$ on $F_{P^*}(M')$.

Proof:

i) $w_1 = v_i \cup \phi - \phi = x^* = x \cup M' \in F_{P^*}(M')$.

ii)
$$\begin{aligned} w_j &= v_i \cup A'_{v_i} - A_{v_i} \\ &= \left(y_i \cup B'_{v_i} \cup A_{v_i}\right) \cup A_{v_j} - A_{v_i} \quad \text{(by Observation 5)} \\ &= y_i \cup B'_{v_i} \cup A'_{v_i} = y_i \cup M'\ .\ \therefore\ w_i \in F_{P^*}(M')\ . \end{aligned}$$

iii) $w_{d^*} = v_{d^*} \cup A'_{v_{d^*}} - A_{v_{d^*}}$

$= y^* \cup A'_{y^*} - A_{y^*}$

$= (y \cup M) \cup M' - M$

$= y \cup M' \;\varepsilon\; F_{P^*}(M')$.

iv) Consider the transition from v_i to v_{i+1} on R^* . There are three possible cases.

<u>Case 1</u>: $v_{i+1} = v_i \cup S - S'$ where $S' \;\varepsilon\; M'$, $S \;\varepsilon\; M$

$$\therefore \; w_{i+1} = v_{i+1} \cup A'_{v_{i+1}} - A_{v_{i+1}}$$

$$= v_{i+1} \cup \left(A'_{v_i} \cup S'\right) - \left(A_{v_i} \cup S\right)$$

$$= (v_{i+1} \cup S' - S) \cup A'_{v_i} - A_{v_i}$$

$$= v_i \cup A'_{v_i} - A_{v_i}$$

$$= w_i \; .$$

$$\therefore \; w_{i+1} = w_i \; .$$

<u>Case 2</u>: $v_{i+1} = v_i \cup S - T$ where $S \;\varepsilon\; P$, $T \;\varepsilon\; P$

$$\therefore \; w_{i+1} = v_{i+1} \cup A'_{v_{i+1}} - A_{v_{i+1}}$$

$$= (v_i \cup S - T) \cup A'_{v_i} - A_{v_i}$$

$$= \left(v_i \cup A'_{v_i} - A_{v_i}\right) \cup S - T$$

$$= w_i \cup S - T \; .$$

$\therefore$ w_i and w_{i+1} are neighbors.

Case 3: $v_{i+1} = v_i \cup S - T'$ where $S \in P$, $T' \in M'$

$$\therefore\ w_{i+1} = v_{i+1} \cup A'_{v_{i+1}} - A_{v_{i+1}}$$

$$= (v_i \cup S - T') \cup \left(A'_{v_i} \cup T'\right) - \left(A_{v_i} \cup T\right)$$

$$= \left(v_i \cup A'_{v_i} - A_{v_i}\right) \cup S - T$$

$$= w_i \cup S - T .$$

$\therefore$ w_i and w_{i+1} are neighbors.

Since R^* is an NR path on P^* , these are the only three possible cases.

v) Immediate from i - iv). ∎

Corollary 2:

∃ an NR path R from x to y on P .

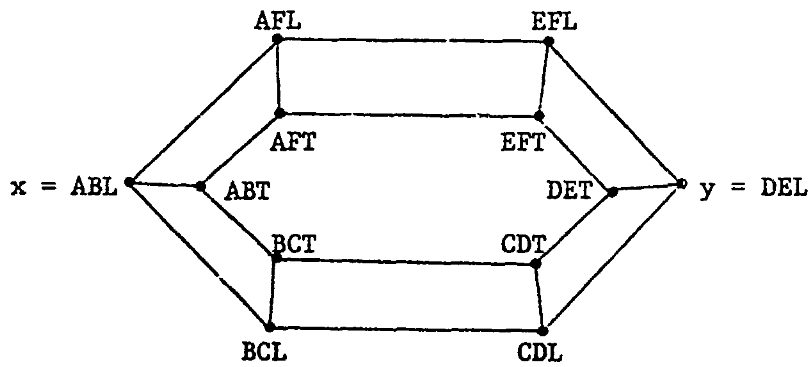
Proof:

Immediate from Lemma 2, Part V, and Observation 3. ∎

4. APPENDIX

Example of Construction 1

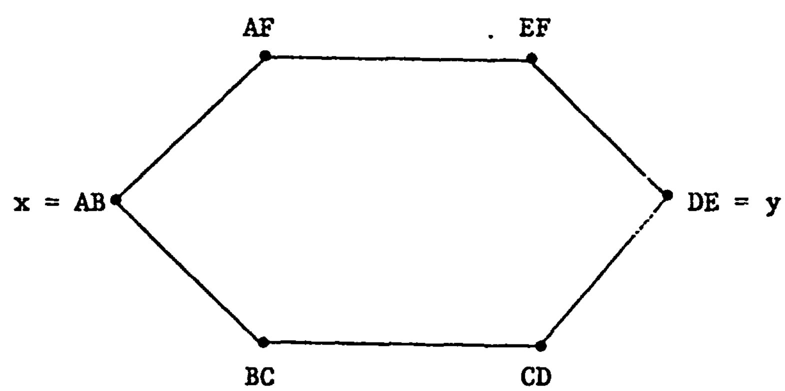
Suppose we have a Polytope P' as follows:



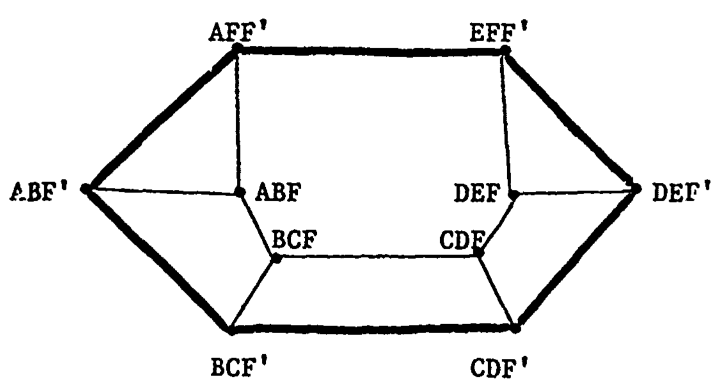
and want NR path from x to y .

Eliminate symbols in common to form face P

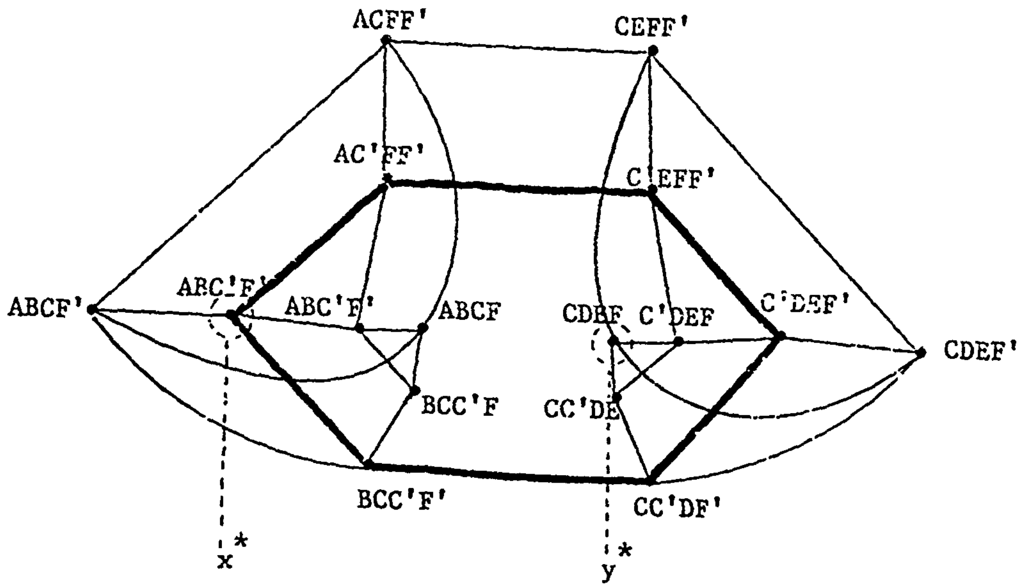
P



Iteration 1: *Choose symbol* F



Iteration 2: *Choose symbol* C



Below are examples of paths R^* from x^* to y^* on P^* and corresponding R from x to y on P according to Construction 2.

(1) R^* : ABC'F' - ABC'F - BCC'F - CC'DF - CDEF

R : AB ———— BC ——— CD ——— DE

(2) R^* : ABC'F' - ABCF' - ACFF' - CEFF' - CDEF

R : AB———————— AF ——— EF ——— DE

(3) R^* : ABC'F' - BCC'F' - CC'DF' - CC'DF - CDEF

R : AB ——— BC ——— CD ———————— DE

(4) R^* : ABC'F' - AC'FF' - C'EFF' - C'DEF - CDEF

R : AB ——— AF ——— EF ——— DE ————

(5) R^* : ABC'F' - BCC'F' - BCC'F - CC'DF - CDEF

R : AB ——— BC ———————— CD ——— DE

(6) R^* : ABC'F' - AC'FF' - ACFF' - CEFF' - CDEF

R : AB ——— AF ———————— EF ——— DE

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[1] Adler, I., "Abstract Polytopes," Ph.D. Thesis, Department of Operations Research, Stanford University, Stanford, California, (1971).

[2] Klee, V. and D. W. Walkup, "The d-Step Conjecture for Polyhedra of Dimension $d < 6$," Acta Mathematics, 117, (1967), 53-78.